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Markov modelling and stochastic identification for nonlinear ship rolling in random waves

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A physically based averaging procedure is applied to a stochastic nonlinear single-degree-of-freedom equation for ship rolling, leading to a one-dimensional continuous Markov model for the energy envelope of the roll motion. It is shown that this model enables various statistics of the roll response to be estimated, including its stationary distribution and the mean time for the energy to reach a critical level. Moreover, it is demonstrated that the Markov model can be used as the basis of a new stochastic identification technique for estimating the spectrum of the excitation, and the nonlinear damping moment, from measurements of the roll response alone.

Keywords: ship rolling; nonlinear; stochastic; identification; damping

1. Introduction

Large roll motions are a serious threat to the safety of a ship and those on board. They may cause excessive loads on sea fastenings, shifting of cargo, shipping of water, loss of men and deck equipment overboard and possibly loss of control of the ship. These factors may contribute to capsize, or to structural failure. An ability to model the rolling motion of a ship in irregular waves is thus a matter of great practical importance. Roll models can be used to predict the motion of a ship in a given sea state. They can also be used in conjunction with roll motion measurements to estimate unknown quantities, such as the damping moment.

Rolling motion is nonlinear in nature and is generally coupled with other motions, such as sway, pitch and heave (see, for example, Nayfeh *et al*. 1973; Thompson *et al*. 1992; Mulk & Falzarano 1994). However, in two special cases it is reasonable to consider this motion as uncoupled and thus governed by a single-degree-of-freedom (SDOF) equation. The first of these is the case of a ship at low speed in unidirectional beam waves. Here, roll is principally coupled with sway and can be uncoupled, to a good approximation, provided that the coordinate origin is located at an appropriate 'roll centre' (Roberts & Dacunha 1985). The second is the case of a ship rolling in unidirectional head waves: in this situation, parametric excitation can lead to loss of stability (Roberts 1982*b*).

In this paper attention is focused on the first of these cases—ship rolling in beam seas. Studies of this problem were initiated by Froude (1861): he obtained an SDOF equation of motion incorporating nonlinear damping and restoring moment terms and from this was able to compute the roll response to regular sinusoidal waves. Subsequent studies of this kind, both analytical and computational, have been made by many workers and a variety of interesting nonlinear phenomena has been observed.

These include the presence of ultraharmonics and subharmonics (Cardo *et al*. 1981; Cardo & Trincas 1987; Peyton Jones & Cankaya 1996) and chaotic motion (Virgin 1987; Thompson *et al*. 1990, 1992). Capsize criteria based on transient response to regular waves (MacMaster & Thompson 1994; Rainey & Thompson 1991; Soliman & Thompson 1991) and on the total energy (Virgin & Erikson 1994) have been proposed.

To complement these deterministic studies, it is important to recognize the random nature of ocean waves and to model both the wave excitation and the response as stochastic processes. Stochastic process theory was first introduced into ship motion studies by St Denis & Pierson (1953), but progress in solving the ship-rolling problem has been slow, due to the unavailability of suitable general methods of dealing with nonlinear systems driven by stochastic processes. Efforts to date include the use of statistical linearization (see, for example, Flower & Mackerdichian 1978), perturbation and functional series (see, for example, Vassilopolos 1967), the Fokker–Planck–Kolmogorov (FPK) equation (Haddara 1974; Moshchuk *et al*. 1995*a*,*b*), non-Gaussian moment closure (Haddara & Zhang 1994), and the use of Melnikov functions (Hsieh *et al*. 1994; Jiang *et al*. 1996).

Of these approaches, the use of the FPK diffusion equation, based on a joint continuous Markov model for the roll displacement and velocity, is one of the most attractive since it enables the probability distribution of the response to be determined. However, the standard theory requires the excitation to be modelled as an ideal white noise process. This is not a realistic assumption since the spectra of ocean waves normally have a distinct peak and a limited bandwidth. An additional complication is that, for an SDOF roll equation of motion, the FPK equation is a two-dimensional partial differential equation. Analytical solutions to this equation are not available for the type of nonlinear damping pertinent to ship roll motion and numerical solutions, while possible, are time consuming to implement.

It has been shown by the first author (Roberts 1982*a*) that these difficulties can be overcome by considering the energy envelope of the response $E(t)$ and applying a stochastic averaging technique, due to Stratonovitch (1963). This approach enables the two-dimensional FPK equation to be reduced to a one-dimensional FPK equation for $E(t)$, containing terms which are explicitly related to the spectrum of the excitation process. This implies that $E(t)$ can be modelled, approximately, as a one-dimensional continuous Markov process. In cases where a stationary distribution exists, at least in an approximate sense, the FPK equation can be solved easily to yield a simple expression for the probability distribution of $E(t)$. Moreover, through a consideration of a related phase process, it is possible to obtain an expression for the stationary joint distribution of the roll displacement and roll velocity. These results have been shown to give a good agreement with simulation results, over a realistic range of damping levels, and also with experimental results obtained using a scale model in a wave tank (Roberts & Dacunha 1985). It is also possible to evaluate, from the FPK equation for $E(t)$, statistics of the first passage type, such as the mean time for the energy to reach a critical level (Roberts 1986*b*). In principal, the probability of capsize, within a specified interval of time, can be estimated by this means.

In the original treatment (Roberts 1982*a*), the reduction to a one-dimensional FPK equation was achieved through the application of the Stratonovitch–Khasminski theorem (see Roberts & Spanos 1986). This required the introduction of an approximation with respect to a phase process which was not consistent with other approximations inherent within the method. In the first part of this paper, an alternative physically based analysis is presented which clarifies the nature of the approximations involved and enables the inconsistency in the earlier treatment to be eliminated. The analysis shows that the expression for the diffusion coefficient in the Markov model for $E(t)$, obtained earlier, is correct, but that a modification to the corresponding expression for the drift coefficient is necessary. An expression for the stationary distribution of $E(t)$ is deduced and compared with the well-known result for white noise excitation. In addition, a method for calculating the mean time to first passage failure is summarized.

In the second part of the paper it is shown that the Markov model can be used for stochastic estimation purposes. For ships at sea the theory given here enables the roll-response statistics to be predicted from a knowledge of the roll-restoring moment, the nonlinear damping moment and the spectrum of the roll-excitation moment. The restoring moment can usually be estimated with reasonable accuracy using hydrostatic theory, but the damping and excitation are much harder to predict. Here it is shown that both the damping moment and the excitation spectrum can be estimated from measurements of the roll response alone, using the Markov model for $E(t)$ as a basis. A great advantage of the proposed new method over earlier treatments (Roberts *et al*. 1992, 1994, 1995, 1996; Roberts & Vasta 1998, 2000*b*; Vasta & Roberts 1998) is that it is not necessary to assume that the response reaches stationarity. The estimation method is validated through application to some digitally simulated data.

2. The equations of motion

It will be assumed here that, at least for the case of beam waves, rolling motion can be treated as uncoupled from other motions and that the roll angle ϕ is governed by a differential equation of motion of the following general form:

$$I\ddot{\phi} + \dot{\phi}C(\phi, \dot{\phi}) + K(\phi) = M(t). \tag{2.1}$$

Here, I is the roll inertia (including added mass), $\dot{\phi}C(\phi, \dot{\phi})$ is the nonlinear damping moment, $K(\phi)$ is the nonlinear restoring moment and $M(t)$ is the roll-excitation moment. $C(\phi, \dot{\phi})$ is assumed to be a positive function of ϕ and $\dot{\phi}$, while $K(\phi)$ is taken to be an anti-symmetric function of ϕ. The excitation moment can, at least in principle, be related to the wave motion, using hydrodynamic theory. For example, Roberts & Dacunha (1985) used linear wave diffraction theory, in combination with Fourier analysis, to generate sample functions of $M(t)$ from corresponding sample functions of measured wave elevation. The equation of motion can be simplified by dividing throughout by I. Thus

$$\ddot{\phi} + \varepsilon^2\dot{\phi}h(\phi, \dot{\phi}) + g(\phi) = \varepsilon x(t), \tag{2.2}$$

where $\varepsilon^2 h = C/I$, $g = K/I$, $\varepsilon x = M/I$.

The scaling parameter ε is introduced into (2.2) to help clarify the order of magnitude of the damping and excitation terms in the subsequent analysis. It will be assumed throughout that the damping is small: hence ε is small. If the response reaches stationary conditions, at least in some approximate sense, then the scaling of the excitation given in (2.2) ensures that its variance remains finite as $\varepsilon \to 0$, i.e. is of order ε^0, with respect to ε (see Roberts & Spanos 1986). If the excitation is not scaled as indicated above, then the variance of the response will be of order ε^{-2}, i.e. it will be become infinitely large as $\varepsilon \to 0$. Thus, in these circumstances, there is no loss of generality implied by scaling the excitation in this way. The scaling simply reflects the fact that, as the response level builds up, under light damping conditions, the contribution of the damping and excitation forces becomes relatively weaker, during a typical cycle in the response, compared with the total energy (kinetic plus potential) in that cycle.

The excitation $x(t)$ will here be modelled as a stationary random process, with zero mean and a power spectrum $S_x(\omega)$ defined by the relation

$$S_x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w_x(\tau)\cos\omega\tau\,\mathrm{d}\tau, \tag{2.3}$$

where

$$w_x(\tau) = E\{x(t)x(t+\tau)\} \tag{2.4}$$

is the covariance function of the excitation and $E\{\cdot\}$ is the expectation, or ensemble averaging operator.

The static restoring moment of a ship $g(\phi)$ is generally of a 'softening' kind such that, as the roll angle increases from zero, it initially increases with ϕ, reaches a maximum value and then falls to zero at some critical angle ϕ^*. In these circumstances, the roll response cannot be treated as a stationary random process since sample functions will eventually reach the critical boundary in the phase plane ϕ, $\dot{\phi}$, corresponding to ϕ^* (see Roberts 1982*a*). The ship will capsize when this boundary is crossed. It follows that normal statistical descriptions of the roll response, such as the standard deviation and the power spectrum, are inappropriate and, instead, one must consider statistics of the 'first passage type', such as the mean time for the response to reach the critical boundary (see, for example, Roberts 1986*a*,*b*). Normally, this mean time will be much larger than a typical roll period: in these circumstances, the response can reach stationarity, in an approximate sense, and the argument given above, with respect to the relative magnitudes of the damping and excitation terms, in terms of the scaling parameter ε, is valid.

The total energy envelope process $E(t)$ associated with the response is defined by

$$E(t) = \tfrac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2.5}$$

where

$$V(\phi) = \int_0^{\phi} g(\xi)\,\mathrm{d}\xi. \tag{2.6}$$

Clearly, $\frac{1}{2}\dot{\phi}^2$ represents the kinetic energy of the ship and $V(\phi)$ is the potential energy. It is possible to rewrite the equation of motion in terms of $E(t)$ and an associated phase process Φ defined by the following relationships:

$$\operatorname{sgn}(\phi)\sqrt{V(\phi)} = \sqrt{E}\cos\Phi, \tag{2.7}$$

$$\dot{\phi} = -\sqrt{2E}\sin\Phi. \tag{2.8}$$

The result is two first-order equations, as follows:

$$\begin{bmatrix} \dot{E} \\ \dot{\Phi} \end{bmatrix} = \begin{bmatrix} -\varepsilon^2 \alpha_1 \\ -\varepsilon^2 \alpha_2 + \gamma \end{bmatrix} + \begin{bmatrix} \varepsilon \beta_1 \\ \varepsilon \beta_2 \end{bmatrix} x(t), \tag{2.9}$$

where

$$\alpha_1 = 2Eh \sin^2 \Phi, \tag{2.10}$$

$$\alpha_2 = -h \sin \Phi \cos \Phi, \tag{2.11}$$

$$\gamma = \frac{|g(x)|}{V(x)}, \tag{2.12}$$

$$\beta_1 = -\sqrt{2E} \sin \Phi, \tag{2.13}$$

$$\beta_2 = -\frac{\cos \Phi}{\sqrt{2E}}. \tag{2.14}$$

3. Markov modelling

If the damping is light, then it is possible to approximate the energy envelope process $E(t)$ as a one-dimensional continuous Markov process. In the theory leading to this approximation, as developed earlier (Roberts 1982*a*), use was made of the Stratonovitch–Khasminskii limit theorem (Roberts & Spanos 1986). However, in order to apply this theorem, it was necessary to make a non-consistent approximation with respect to the γ term, for which there is no theoretical foundation.

Recently, an alternative approach to the derivation of a Markov model has been outlined (Roberts & Vasta 2000*a*), based on physical reasoning, which overcomes this difficulty and leads to a correction to previous results (Roberts 1982*a*). Here, this approach is fully developed.

(*a*) *The Markov process approximation*

A fundamental assumption, in developing a Markov model, is that it is possible to find an interval of time Δt, such that $\Delta t > \tau_{\text{cor}}$, where τ_{cor} is the correlation time-scale of the excitation, but small enough that the change of energy $\Delta E = E(t + \Delta t) - E(t)$, in that interval, is relatively small. This condition can be met if the damping is sufficiently light: as the bandwidth of the excitation increases, τ_{cor} becomes smaller and the restriction on the damping level reduces.

If the coefficients defined by

$$H_n(E) = \frac{1}{\Delta t} E\{\Delta E^n\}, \quad n = 1, 2, \ldots, \tag{3.1}$$

are such that $H_n(E) = 0$, for $n > 2$, and both

$$m(E) \equiv H_1(E), \tag{3.2}$$

and

$$D(E) \equiv H_2(E) \tag{3.3}$$

approach a limiting value, for small Δt (subject to $\Delta t > \tau_{\text{cor}}$), then $E(t)$ can be modelled, approximately, as a Markov process (see, for example, Gardiner 1985).

The coefficients $m(E)$ and $D(E)$ are known, respectively, as the drift and diffusion coefficients.

This Markov process is governed by the following stochastic Itô equation:

$$\mathrm{d}E = m\,\mathrm{d}t + D^{1/2}\,\mathrm{d}W, \tag{3.4}$$

where W is a unit Wiener (or Brownian) process. The process is also defined by its transition density function $p(E \mid E_0; t)$, where $p(E \mid E_0; t)\mathrm{d}E$ is the probability that the process lies in the range E to $E + \mathrm{d}E$, at time t, given that it was at E_0 at time $t = 0$. From (3.4), it follows that $p(E \mid E_0; t)$ is governed by the Fokker–Plank–Kolmogorov (FPK) equation (Gardiner 1985),

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial E}(mp) + \frac{1}{2}\frac{\partial^2}{\partial E^2}(Dp). \tag{3.5}$$

(b) *Averaging the dissipation terms*

If s is some time in the interval t to $t + \Delta t$, then one can write

$$E(s) = E_0 + e(s), \tag{3.6}$$

$$\Phi(s) = \Phi_0(s) + \theta(s), \tag{3.7}$$

where E_0, Φ_0 are the free undamped solutions ($\varepsilon = 0$). From (2.9),

$$\dot{E}_0 = 0, \tag{3.8}$$

$$\dot{\Phi}_0 = \gamma(E_0, \Phi_0) \equiv \gamma_0, \tag{3.9}$$

and hence

$$E_0 = \text{const.} \tag{3.10}$$

and

$$\Phi_0 = \int \gamma_0\,\mathrm{d}t. \tag{3.11}$$

For the special case of a linear restoring moment, γ_0 is a constant, equal to the natural undamped frequency of oscillation, and Φ_0 increases linearly with time.

As a first step in developing the Markov model, the dissipation terms α_1 and α_2 in (2.6) can be averaged over the period of free undamped oscillation $T(E)$, treating E as a constant during this period and setting $\Phi = \Phi_0$. This corresponds to the well-known Krylov–Bogoliubov averaging method used in deterministic nonlinear problems (Bogoliubov & Mitropolsky 1961) and is correct to order ε^2. The equations of motion then become

$$\begin{bmatrix} \dot{E} \\ \dot{\Phi} \end{bmatrix} = \begin{bmatrix} -\varepsilon^2 \Lambda_1 \\ -\varepsilon^2 \Lambda_2 + \gamma \end{bmatrix} + \begin{bmatrix} \varepsilon\beta_1 \\ \varepsilon\beta_2 \end{bmatrix} x(t), \tag{3.12}$$

where

$$\Lambda_1(E) = \frac{2E}{T(E)} \oint h \sin^2\Phi_0\,\mathrm{d}t, \tag{3.13}$$

$$\Lambda_2(E) = -\frac{1}{T(E)} \oint h \sin\Phi_0 \cos\Phi_0\,\mathrm{d}t. \tag{3.14}$$

In the subsequent analysis, only the first of these damping functions, $\Lambda_1(E)$, is required. This function can be expressed in terms of the Fourier expansion of $\sin\Phi_0$ as follows:

$$\sin\Phi_0 = \sum_{n=1,3,\dots}^{\infty} s_n \sin n\omega(E)t. \tag{3.15}$$

Hence

$$\Lambda_1(E) = \frac{2E}{T(E)} \sum_{n=1,3,\dots}^{\infty} s_n^2 \oint h \sin^2 n\omega(E)t \,\mathrm{d}t. \tag{3.16}$$

In most cases, the damping function $h(\phi,\dot\phi)$ can be modelled in a simple linear-in-the-parameters form. The integral in (3.16) can then be evaluated quite easily. Unless the nonlinearity with respect to the restoring moment is very severe, the series can be truncated at $n = 1$, to a good level of approximation.

In the particular case where $h(\phi,\dot\phi)$ depends only on the energy level E, i.e.

$$h(\phi,\dot\phi) = h(E), \tag{3.17}$$

then, from (3.16),

$$\Lambda_1(E) = h(E)s(E)E \approx h(E)s_1^2 E, \tag{3.18}$$

where

$$s(E) = \sum_{n=1,3,\dots}^{n} s_n^2 = \frac{2}{T(E)} \oint \sin^2\Phi_0(t)\,\mathrm{d}t. \tag{3.19}$$

For a linear restoring moment, $s(E) = 1$ and (3.18) thus reduces to

$$\Lambda_1(E) = h(E)E.$$

If the damping is of the linear viscous form, such that $h(\phi,\dot\phi) = \beta$, where β is the damping coefficient, then (3.18) gives $\Lambda_1(E) = \beta s(E)E$: this reduces to $\Lambda_1(E) = \beta E$ if the restoring force is also linear.

The period of free undamped oscillation is given by

$$T(E) = \frac{2\pi}{\omega(E)} = 2\sqrt{2}\int_0^b \frac{\mathrm{d}\xi}{\sqrt{[E - V(\xi)]}}, \tag{3.20}$$

where b is such that $V(b) = E$ and $\omega(E)$ is the corresponding frequency.

(c) *Evaluation of the drift coefficient*

On integrating the first row of (3.12) over the interval t to $t + \Delta t$, and using the definition of the drift coefficient given by (3.1) and (3.2), one obtains the expression

$$m(E) = -\varepsilon^2\Lambda_1(E) + \frac{\varepsilon}{\Delta t}\int_t^{t+\Delta t} E\{\beta_1 x\}\,\mathrm{d}u. \tag{3.21}$$

The integral term is non-zero, due to the correlation between β_1 and x.

To evaluate the integral, β_1 can be expanded about its deterministic evolution, corresponding to $\varepsilon = 0$, denoted $\beta_{1,0} = \beta(E_0, \Phi_0)$. Thus, for $t < u < t + \Delta t$,

$$\beta_1(u) = \beta_{1,0}(u) + \left(\frac{\partial \beta_1}{\partial E}\right)_0^u e(u) + \left(\frac{\partial \beta_1}{\partial \Phi}\right)_0^u \theta(u). \tag{3.22}$$

Here, terms of higher order than unity, with respect to e and θ, have been neglected. The expansion is thus correct to order ε. On substituting this expansion into (3.21), and using the fact that the contribution of the first term is zero (since $E\{x(t)\} = 0$), one obtains

$$m(E) = -\varepsilon^2 \Lambda_1(E) + I_1 + I_2, \tag{3.23}$$

where

$$I_1 = \frac{\varepsilon}{\Delta t} \int_t^{t+\Delta t} \left(\frac{\partial \beta_1}{\partial E}\right)_0^u E\{e(u)x(u)\}\, \mathrm{d}u, \tag{3.24}$$

$$I_2 = \frac{\varepsilon}{\Delta t} \int_t^{t+\Delta t} \left(\frac{\partial \beta_1}{\partial \Phi}\right)_0^u E\{\theta(u)x(u)\}\, \mathrm{d}u. \tag{3.25}$$

(i) *Evaluation of* I_1

To evaluate the integral I_1, an expression for $e(u)$ can be obtained by integrating the energy equation (first row of (3.12)). Correct to order ε, one has

$$e(u) = \varepsilon \int_t^u \beta_{1,0}(v) x(v)\, \mathrm{d}v. \tag{3.26}$$

Hence, on combining (3.24) and (3.26), one has

$$I_1 = \frac{\varepsilon^2}{\Delta t} \int_t^{t+\Delta t} \int_t^u \left(\frac{\partial \beta_1}{\partial E}\right)_0^u \beta_{1,0}(v) E\{x(u)x(v)\}\, \mathrm{d}v\, \mathrm{d}u. \tag{3.27}$$

Using the definition of the correlation function for $x(t)$ given by (2.4), together with the explicit expression for β_1 given by (2.13), equation (3.27) becomes

$$I_1 = \frac{\varepsilon^2}{\Delta t} \int_t^{t+\Delta t} \int_t^u \sin \Phi_0(u) \sin \Phi_0(v) w_x(u - v)\, \mathrm{d}v\, \mathrm{d}u. \tag{3.28}$$

Combining (3.15) and (3.28), one obtains

$$I_1 = \tfrac{1}{2} \varepsilon^2 \pi \sum_{n=1,3,\dots}^{\infty} s_n^2 S_x[n\omega(E)]. \tag{3.29}$$

(ii) *Evaluation of* I_2

To evaluate the integral I_2, it is necessary to obtain an expression for θ. As a first step, the term γ (see equation (2.12)) can be expanded about its value when excitation and damping are absent, in a similar way to the method used for β_1. Thus

$$\gamma(t) = \gamma_{1,0}(t) + \gamma_0^E e(t) + \gamma_0^\Phi \theta(t), \tag{3.30}$$

where

$$\gamma_0^E(t) = \left(\frac{\partial\gamma}{\partial E}\right)_0^t, \tag{3.31}$$

$$\gamma_0^\Phi(t) = \left(\frac{\partial\gamma}{\partial \Phi}\right)_0^t. \tag{3.32}$$

Again, this expansion is correct to order ε. On substituting this expansion into the phase equation (second row of (3.12)), one obtains the following first-order differential equation for θ:

$$\dot{\theta} - \gamma_0^\Phi(t)\theta = -\varepsilon^2 \Lambda_1(E) + \varepsilon\beta_2 x(t) + \gamma_0^E e(t). \tag{3.33}$$

An integration of this equation yields, correct to order ε,

$$\theta(u) = \varepsilon \int_0^u \beta_{2,0}(v)h(u,v)\,\mathrm{d}v + \varepsilon \int_t^u \int_t^v \gamma_0^E(v)h(u,v)\beta_{1,0}(\xi)x(\xi)\,\mathrm{d}\xi\,\mathrm{d}v. \tag{3.34}$$

Here, $h(u,v)$ is the impulse response function for differential equation and is given by

$$h(u,v) = \frac{g(v)}{g(u)}, \tag{3.35}$$

where

$$g(t) = \exp\left(-\int \gamma_0^\Phi(u)\,\mathrm{d}u\right). \tag{3.36}$$

On combining (3.34) and (3.35) and substituting the resulting expression for θ into (3.25), one obtains

$$I_2 = J_1 + J_2, \tag{3.37}$$

where

$$J_1 = \frac{\varepsilon^2}{\Delta t}\int_t^{t+\Delta t}\int_t^u F(u)G(v)w(u-v)\,\mathrm{d}v\,\mathrm{d}u, \tag{3.38}$$

$$J_2 = \frac{2\varepsilon^2 E}{\Delta t}\int_t^{t+\Delta t}\int_t^u\int_t^v F(u)\sin\Phi_0(\xi)\kappa(v)w_x(u-\xi)\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}u \tag{3.39}$$

and

$$F(t) = \frac{\cos\Phi_0(t)}{g(t)}, \tag{3.40}$$

$$G(t) = \cos\Phi_0(t)g(t), \tag{3.41}$$

$$\kappa(t) = \gamma_0^E(t)g(t). \tag{3.42}$$

The integral J_1 can be simplified by expanding $F(t)$ and $G(t)$ as Fourier series, as follows:

$$F(u) = \sum_{n=1,3,\dots}^{\infty} c_n^b \cos[n\omega(E)t], \tag{3.43}$$

$$G(u) = \sum_{n=1,3,\dots}^{\infty} c_n^t \cos[n\omega(E)t]. \tag{3.44}$$

Substituting these expansions into (3.38), one finds that

$$J_1 = \tfrac{1}{2}\varepsilon^2\pi \sum_{n=1,3,\dots}^{\infty} c_n^t c_n^b S_x[n\omega(E)]. \tag{3.45}$$

Using the Fourier expansions given by (3.15) and (3.44), together with the expression for $h(u,v)$, the integral J_2 defined by (3.39) can be expressed as follows:

$$J_2 = \frac{\varepsilon^2 E}{\Delta t}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(A_{nm}+B_{nm}), \tag{3.46}$$

where

$$A_{nm} = c_n^b s_m \int_{-\infty}^{\infty} S_x(\omega)\,\mathrm{d}\omega \int_{\Re} \sin[(nu+m\xi)\omega(E)]\kappa(v)\cos\omega(u-\xi)\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}u, \tag{3.47}$$

$$B_{nm} = c_n^b s_m \int_{-\infty}^{\infty} S_x(\omega)\,\mathrm{d}\omega \int_{\Re} \sin[(nu-m\xi)\omega(E)]\kappa(v)\cos\omega(u-\xi)\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}u. \tag{3.48}$$

Here, $\Re$ is the integration space defined by (3.39) and the relationship between the correlation function of the power spectrum of the excitation process has been used. As the integration volume, controlled by Δt, increases, the oscillatory nature of the integrands in (3.47) and (3.48), with respect to u, ξ and v, is such that A_{nm} and B_{nm} become negligibly small for all n, m.

(iii) *Complete results*

On collecting results, the following expression for the drift coefficient, correct to order ε^2, is obtained:

$$m(E) = -d(E) + \tfrac{1}{2}\pi \sum_{n=1,3,\dots}^{\infty} (s_n^2 + c_n^t c_n^b) S_y[n\omega(E)], \tag{3.49}$$

where $d(E)$ is the average of the actual damping moment and $y(t) = \varepsilon x(t)$ is the actual excitation process. Thus

$$d(E) = \varepsilon^2 \Lambda_1(E), \tag{3.50}$$

$$S_y(\omega) = \varepsilon^2 S_x(\omega). \tag{3.51}$$

In the analysis given earlier (Roberts 1982*a*), the terms γ_0^E and γ_0^{Φ} were neglected. Then $g(t) = 1$ and $c_n^t = c_n^b = c_n$, where

$$\cos\Phi(t)_0 = \sum_{n=1,3,\dots}^{\infty} c_n \cos[n\omega(E)t]. \tag{3.52}$$

A parametric study by the authors, for the case of a Duffing oscillator, has revealed that the error involved in using c_n^2 in place of $c_n^t c_n^b$ is usually small (Roberts & Vasta 2000*a*).

In nearly all cases, the first term in the Fourier expansion is dominant. One can then, to a very good approximation, write

$$m(E) = -d(E) + \tfrac{1}{2}\pi(s_1^2 + c_1^t c_1^b) S_y[n\omega(E)]. \tag{3.53}$$

(iv) *White noise excitation*

In the special case where the excitation can be modelled as a white noise, we have

$$w_y(\tau) = I_y \delta(\tau), \tag{3.54}$$

$$S_y(\omega) = \frac{I_y}{2\pi} \equiv S_{y0}; \tag{3.55}$$

then (3.49) reduces to

$$m(E) = -d(E) + \tfrac{1}{2}\pi S_{y0} \sum_{n=1,3,\dots}^{\infty} (s_n^2 + c_n^2) = d(E) + \pi S_{y0}. \tag{3.56}$$

This result was obtained earlier by Stratonovitch (1963), using a different argument. It is interesting to note that, in this case, the second term in this expression is related to the mean rate of energy fed into the system (Roberts 1983), which is independent of the form of the nonlinear stiffness.

(*d*) *Evaluation of the diffusion coefficient*

The diffusion coefficient is much easier to calculate than the drift coefficient. From (3.12), one finds immediately that, correct to order ε^2,

$$\begin{aligned} D(E) &= \frac{\varepsilon^2}{\Delta t} \int_t^{t+\Delta t} \int_t^{t+\Delta t} \beta_{1,0}(u)\beta_{1,0}(v) w(u-v)\,\mathrm{d}u\,\mathrm{d}v \\ &= \frac{\varepsilon^2 2E}{\Delta t} \int_t^{t+\Delta t} \int_t^{t+\Delta t} \sin \Phi_0(u) \sin \Phi_0(v) w(u-v)\,\mathrm{d}u\,\mathrm{d}v. \end{aligned} \tag{3.57}$$

On averaging over time, and using the Fourier series expansion of $\sin \Phi_0$ again, it is found that

$$D(E) = 2\pi E \sum_{n=1,3,\dots}^{\infty} s_n^2 S_y[n\omega(E)]. \tag{3.58}$$

Again, in nearly all cases the first term in the Fourier expansion is dominant. Then, to a very good approximation, one can write

$$D(E) = 2\pi E s_1^2 S_y[\omega(E)]. \tag{3.59}$$

(i) *White noise excitation*

In the particular case of white noise excitation, equation (3.59) reduces to

$$D(E) = 2\pi S_{y0} E s(E). \tag{3.60}$$

Again, this result was obtained earlier by Stratonovitch (1963).

4. Predicting the response

(*a*) *Stationary response*

The probability density function for the stationary energy envelope response $p(E)$ can be found, when it exists, by solving (3.5) with $\partial p/\partial t = 0$. As pointed out earlier, a stationary response is not achievable if the restoring force is of the softening kind, but will exist, in an approximate sense, if the mean time to capsize is very long. In these circumstances, it is reasonable to model the restoring force in terms of a non-softening characteristic, for small to moderate roll angles, and to consider the stationary distribution of the response.

Using an argument similar to that given by Krenk & Roberts (1999), $p(E)$ can be written as

$$p(E) = \frac{CT(E)}{\pi S^*(E)} \exp\left(- \int_0^E \frac{h^*(\xi)}{\pi S^*(\xi)} \, d\xi \right), \tag{4.1}$$

where C is a normalization constant and $S^*(E)$ is an 'effective spectral density', defined by

$$S^*(E) = \sum_{n=1,3,\dots}^{\infty} \frac{s_n^2}{s(E)} S_y[n\omega(E)]. \tag{4.2}$$

Also,

$$h^*(E) = h_{\text{eq}}(E) + \frac{\pi}{s(E)E} \sum_{n=1,3,\dots}^{\infty} \left(\frac{s_n^2}{s(E)} - \tfrac{1}{2}(s_n^2 + c_n^t c_n^b) \right) S_y[n\omega(E)], \tag{4.3}$$

where $h_{\text{eq}}(E)$ is an 'effective damping function', defined by

$$h_{\text{eq}} = \frac{d(E)}{s(E)E}. \tag{4.4}$$

By referring to (3.18), it can be seen that $h_{\text{eq}}(E) = \varepsilon^2 h(\phi, \dot{\phi})$ in the case where the latter depends only on E. These expressions reduce to the result given by Krenk & Roberts (1999) if $c_n^t = c_n^b = c_n$.

Moreover, using a similarity argument (Krenk & Roberts 1999), one obtains

$$p(\phi, \dot{\phi}) = \frac{p(E)}{T(E)}. \tag{4.5}$$

In the particular case of white noise excitation, $h^* = h_{\text{eq}}$ and $S^* = S_0$. Hence (4.1) reduces to

$$p(E) = \frac{CT(E)}{\pi S_{y0}} \exp\left(- \int_0^E \frac{h_{\text{eq}}(\xi)}{\pi S_{y0}} \, d\xi \right). \tag{4.6}$$

This result coincides with the result found earlier by Stratonovitch (1963), using a generalized form of stochastic averaging. When the damping is of the form given by (3.17), then the result is exact (see Caughey 1971).

Some comparisons between simulation estimates of $p(E)$ and corresponding theoretical estimates, derived from the present theory, for the case of a Duffing oscillator with linear damping, have been presented recently (Roberts & Vasta 2000*a*). These results show that the correction to the original theory, given here, is small and likely to be negligible in most cases.

(b) First passage statistics

A great advantage of the Markov modelling approach is that it allows one to obtain response statistics of the first passage type.

The first passage time for $E(t)$ is simply the time $T(E_0)$ taken for a sample function of the process, starting at E_0 at time $t = 0$, to reach some critical level, h. The density function of this time is denoted here by $p(E_0;t)$. For capsize, $h = V(\phi^*)$.

For long times to failure, the first passage density function takes the asymptotic form (Roberts 1986*a*)

$$p(E_0;t) = \lambda_1 \exp(-\lambda_1 t), \tag{4.7}$$

where λ_1 is the first eigenvalue of the adjoint of the FPK equation for $E(t)$ (see equation (3.5)). If $p(E_0;t)$ obeys this asymptotic form, then

$$M_1(E_0) = \int_0^\infty \lambda_1 \exp(-\lambda_1 t) t \, \mathrm{d}t = \frac{1}{\lambda_1}, \tag{4.8}$$

where M_1 is the first moment, or mean, of the first passage time. It follows that a knowledge of M_1 alone suffices to estimate the first passage density function and hence the probability that $E(t)$ stays within prescribed limits, within a fixed interval of time.

An ordinary differential equation for M_1 can be obtained from the adjoint FPK equation for $E(t)$ (see Roberts 1986*a*,*b*). Thus

$$-1 = m(E_0)\frac{\mathrm{d}M_1}{\mathrm{d}E_0} + \tfrac{1}{2}D(E_0)\frac{\mathrm{d}^2 M_1}{\mathrm{d}E_0^2}. \tag{4.9}$$

Appropriate boundary conditions have been given by Roberts (1986*b*). An analytic solution to this equation is generally difficult to find but numerical solutions are easily obtainable.

A comparison between theoretical estimates of $\bar{T} \equiv M(0)$ and corresponding simulation results has been discussed by Roberts (1986*b*). Here, an SDOF system with linear-plus-quadratic damping and a softening restoring characteristic (linear-minus-cubic), driven by non-white excitation, was considered and estimates of $\bar{T}$ were plotted against a parameter defining the bandwidth of the excitation. It was observed that the simulation estimates approach the theoretical estimates as the bandwidth of the excitation increases but that this approach is slow. At bandwidths typical of those encountered in practical applications the theory underestimates $\bar{T}$ quite significantly, i.e. the theoretical estimate is conservative and thus may be useful for design purposes. The discrepancy for small excitation bandwidths is due to the fact that sample functions of the actual $E(t)$ process are smoother, in these circumstances, than those generated by the corresponding Markov model. The latter are highly irregular, on a microscopic scale, and the mean time to failure is very sensitive to small-scale irregularities.

5. Stochastic system identification

The theory given earlier in this paper can also be used for stochastic estimation purposes. It is very difficult to predict the damping moment experienced by a ship at sea by theoretical methods based on hydrodynamics, due to the complex three-dimensional and time-dependent nature of the flow involved. For similar reasons, the wave excitation moment is very difficult to relate to wave motion. However, at sea, measurements of the variation of the roll angle with time can be obtained fairly readily. It is therefore natural to enquire if the damping and excitation moments can be estimated from a time history of the roll angle alone. To achieve this, it is, of course, necessary to model the excitation moment in some way. If the excitation is unmeasurable, then it can be treated as a stationary stochastic process. One is then faced with a stochastic identification problem, which can be stated as follows: from a single measured time history of the roll angle $\phi(t_i) \equiv \phi(i\delta t)$ $(i = 1, 2, \dots, N)$, where δt is the sampling interval, together with a stationary stochastic process model of the excitation, how can one generate estimates of (a) the damping moment and (b) the power spectrum of the excitation?

Some progress towards solving this problem has already been made (Roberts *et al*. 1992, 1994, 1995, 1996; Roberts & Vasta 1998, 2000*b*; Vasta & Roberts 1998). However, these studies suffer from two basic limitations. Firstly, the excitation was assumed, *a priori*, to have a specific parametric form (usually white noise): estimation of the excitation thus consisted of determining the parameters defining this process. Secondly, to apply the proposed methods, it was necessary to assume that the roll response could be treated as a stationary stochastic process. As already pointed out, in the case of ship rolling, the restoring moment is of the softening kind, such that response stationarity cannot be achieved. Thus the response will eventually reach the critical angle corresponding to capsize. The mean time to capsize will clearly increase as the level of excitation reduces, and for sufficiently low levels it may be possible to treat the response as stationary but only in some approximate sense which is difficult to define.

In this paper an entirely new method of solving the stochastic estimation problem is proposed, which overcomes these two limitations. It is based on the theory developed in § 2 and, in particular, on the approximate theoretical expressions obtained there for the drift coefficient (see equation (3.53)) and the diffusion coefficient (see equation (3.59). Both these coefficients can be estimated directly by suitably processing the measured roll-angle data $\phi(t_i)$, using the definitions given by (3.1)–(3.2).

(*a*) *Estimation method*

As a first stage, the roll-angle data are converted to energy samples $E(t_i)$, using the definition of E given by (2.5) and (2.6). This is clearly straightforward if the restoring moment $g(\phi)$ is known. However, a method for doing this, which does not require a knowledge of $g(\phi)$, is also available (Roberts *et al*. 1992). Then an interval of time, $\Delta t = m\delta t$, is chosen and the increments

$$\Delta E_i = E(t_i) - E(t_{i-m}), \quad i = m + 1, \dots, N, \tag{5.1}$$

are computed. Following this, the energy range is divided up into a number of contiguous 'slots', each of width δE, such that the jth slot covers the energy range $(j - 1)\delta E$ to $j\delta E$ $(j = 1, 2, \dots)$. Each ΔE_i is assigned to the slot for which

$$j\delta E < E(t_{i-m}) < (j + 1)\delta E.$$

Estimates of the drift and diffusion coefficients can then be generated by averaging the ΔE_i values and their square, within each slot. Thus

$$\hat{m}(E_j) = \frac{1}{n_j \Delta t} \sum_{(j-1)\delta E < E(t_{i-m}) < j\delta E} \Delta E_i, \tag{5.2}$$

$$\hat{D}(E_j) = \frac{1}{n_j \Delta t} \sum_{(j-1)\delta E < E(t_{i-m}) < j\delta E} (\Delta E_i)^2, \tag{5.3}$$

where E_j is the mid-value of E in the jth slot and n_j is the number of ΔE_i values obtained for the jth slot.

For identification purposes, a correspondence between these estimates and the theoretical values, given by (3.53) and (3.59), is required: this will clearly depend on the choice of Δt. If Δt is too small, then $\Delta t < \tau_{\text{cor}}$, contravening one of the basic assumptions in the theory. On the other hand, if Δt is too large, then the ΔE_i will become large, whereas the theory assumes small changes in energy level over the interval Δt. For lightly damped systems, at least, one can expect the estimates of the coefficients, given by (5.2) and (5.3), to be almost independent of Δt, over a significant 'stable' range of Δt values, and to correspond well with the theoretical expressions, within this range. The correlation time-scale of the excitation, τ_{cor}, will not be known in practice, but one can evaluate the drift and diffusion coefficients for a range of Δt and choose a Δt value within the stable range.

Using (3.59), it is possible to generate an estimate of the power spectrum of the excitation process $y(t)$ directly from $\hat{D}(E_j)$. Thus

$$\hat{S}_y(\omega_j) = \frac{\hat{D}(E_j)}{2\pi E_j s_1^2}, \tag{5.4}$$

where

$$\omega_j = \omega(E_j). \tag{5.5}$$

It is noted that in order to evaluate s_1^2, a knowledge of the restoring characteristic is required. This can either be calculated from hydrostatics or estimates directly from the data, using the method described by Roberts *et al.* (1992). If the restoring characteristic is linear, or approximately so, over the roll amplitude range covered by the data, then $s_1^2 \approx 1$ and a knowledge of the linear stiffness coefficient is not required. In this case, $\omega_j = \omega_0$, independent of E_j, where ω_0 is the undamped natural frequency.

Two aspects of the power spectrum estimate given by (5.4) are worth noting. Firstly, the estimation of this quantity has been totally *uncoupled* from the estimation of the damping moment. In earlier attempts to use a Markov model (Roberts *et al.* 1992, 1996), based on the probability density of an assumed stationary response to white noise excitation, only the ratio of the damping moment to the excitation moment could be determined (this can seen by inspection of (4.6)). Secondly, this estimate is *non-parametric*, i.e. it is not necessary to assume a parametric form.

Once $\hat{S}_y(\omega_j)$ is determined, the damping moment can be estimated using (3.53). Thus

$$\hat{d}(E_j) = -\hat{m}(E_j) + \tfrac{1}{2}\pi(s_1^2 + c_1^t c_1^b)\hat{S}_y(\omega_j). \tag{5.6}$$

Again, to evaluate the Fourier coefficients in this expression, a knowledge of the restoring characteristic is needed, in general. However, there are two important special cases where this information is not required. Firstly, if the excitation is a white noise, or approximately so, then, from (3.56),

$$\hat{d}(E_j) = -\hat{m}(E_j) + \pi \hat{S}_{y0}. \tag{5.7}$$

Secondly, if the restoring characteristic is linear, or approximately so, then (5.6) reduces to

$$\hat{d}(E_j) = -\hat{m}(E_j) + \pi \hat{S}_y(\omega_0). \tag{5.8}$$

Evidently, in this case, the excitation can be modelled as an equivalent white noise, with level $S_{y0} = S_y(\omega_0)$.

In practice, there will usually be some noise on the response data, arising from the measuring instrumentation. In the present work, it is assumed that the power spectrum of such noise is in a frequency range which is much higher than the dynamic range of the ship roll motion. In these circumstances, it is a simple matter to preprocess the response data by applying a low-pass digital filter.

The accuracy of the estimates obtained by the proposed estimation scheme will clearly depend on the record length. As is the usual case in statistical analysis, one can expect that the variance of the estimates will be inversely proportional to the square root of the record length.

(*b*) *An example*

To illustrate the method, through application to some simulation data, a particular form of (2.2) is chosen, where

$$\ddot{\phi} + A\dot{\phi} + B\dot{\phi}|\dot{\phi}| + k_1\phi - k_2\phi^3 = y(t). \tag{5.9}$$

Thus the damping is assumed to be of the linear-plus-quadratic type. It has been shown by numerous studies of experimental data (see, for example, Roberts 1985; Gawthrop *et al.* 1988; Kountzeris *et al.* 1990; Roberts *et al.* 1991) that this is a good model. The linear-minus-cubic form chosen for the restoring moment is the simplest model that represents the actual shape of measured restoring moment versus roll-angle curves.

On non-dimensionalizing time with respect to the linear undamped natural frequency $\omega_0 = \sqrt{k_1}$, and the roll angle with respect to the critical angle $\phi^* = \sqrt{k_1/k_2}$, equation (5.9) can be recast as

$$\ddot{\phi} + a\dot{\phi} + b\dot{\phi}|\dot{\phi}| + \phi - \phi^3 = y(t). \tag{5.10}$$

For the purpose of testing the estimation method, it is not necessary to model the excitation process in terms of a spectrum with a standard form. It is sufficient to use a model with approximately the right characteristics: in this case, a spectrum with a single well-defined peak.

The excitation process considered here can be obtained by filtering white noise $n(t)$ through two identical first-order linear filters, placed in series. The output $\eta(t)$ is obtained from

$$\dot{\eta} + \beta\eta = \beta\lambda(t), \tag{5.11}$$

$$\dot{\lambda} + \beta\eta = \beta n(t). \tag{5.12}$$

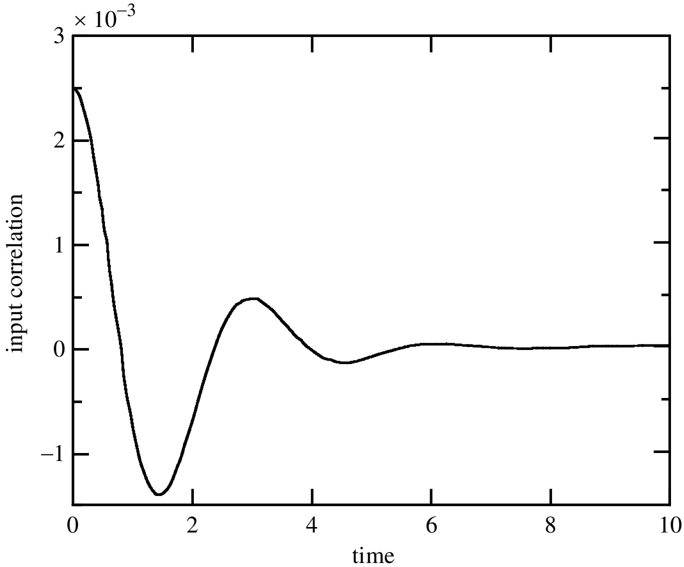

Figure 1. The correlation function of the excitation.

If two such filtering operations are applied to independent white noises, the resulting outputs, $\eta_1(t)$ and $\eta_2(t)$, may be combined to form the input process

$$y(t) = h[\eta_1(t)\cos\omega_\mathrm{p}t + \eta_2(t)\sin\omega_\mathrm{p}t]. \tag{5.13}$$

The power spectrum of $y(t)$, so defined, is given by

$$S_y(\omega) = k\left[\left\{\frac{1}{\beta^2 + (\omega - \omega_\mathrm{p})^2}\right\}^2 + \left\{\frac{1}{\beta^2 + (\omega + \omega_\mathrm{p})^2}\right\}^2\right], \tag{5.14}$$

where k is a scaling constant. For small β, this spectrum has a single peak in the neighbourhood of the frequency ω_p. The parameter β controls the bandwidth of the excitation while, ω_p determines the position of the peak, relative to the linear undamped natural frequency ($\omega = 1$).

It is worth emphasizing that the proposed method is not limited to input spectra of the kind discussed here, for illustrative purposes. It is applicable to any wave excitation, provided that this can be modelled as a stationary random process.

To test the theory, sample functions of the roll-angle response were generated. These were achieved by first generating two sequences of Gaussian independent random numbers, using a pseudo-random number generator, to simulate sample functions of two independent white noise processes. These were then filtered, and multiplied by sinusoidal functions, according to (5.11)–(5.13), to produce a sample function of the excitation process. Finally, the equation of motion was solved numerically, using the fourth-order Runge–Kutta algorithm, to generate a sample function of ϕ.

As an initial test, a case where the damping is linear and light ($a = 0.01$) was chosen. This is referred to henceforth as case 1. A sample function of ϕ, $\phi(t_i)$, was computed, where $t_i = i\delta t$, $i = 1, 2, \ldots, N$, $N = 100\,000$ and $\delta t = 0.05$. This gives a record length of 5000, containing approximately 1000 roll cycles. This is a typical amount of data that could be collected in practice during a period of time in

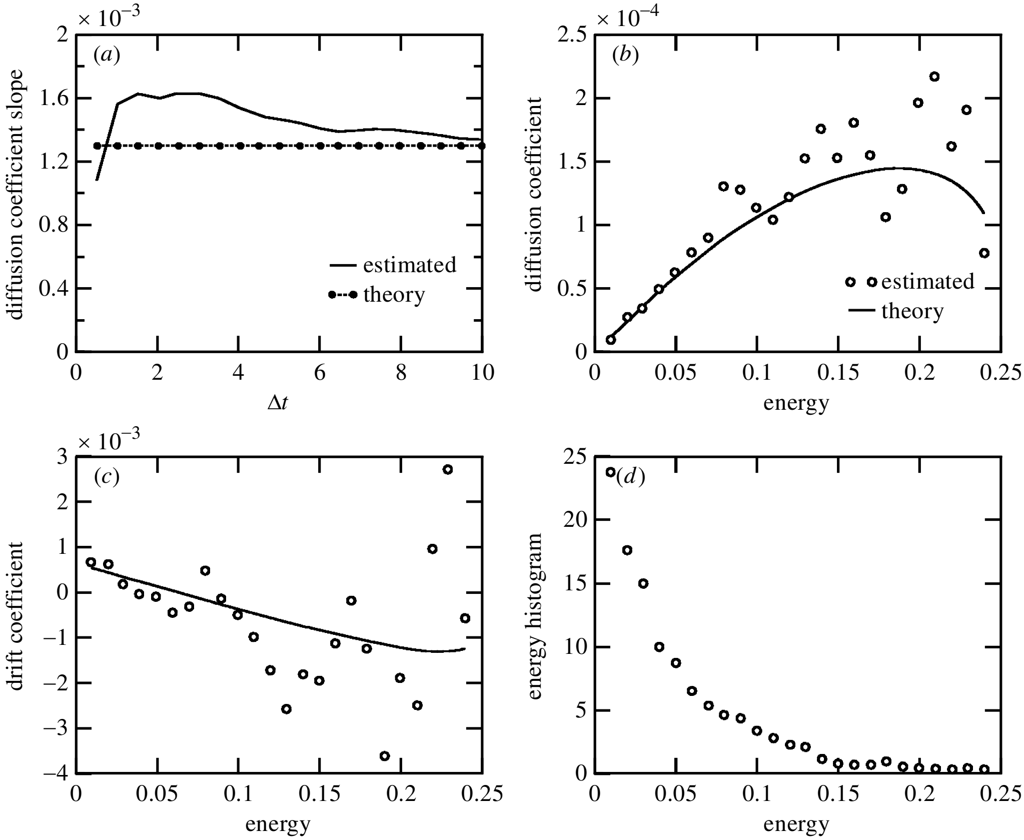

Figure 2. (*a*) Variation of the slope of the estimated diffusion coefficient with the interval Δt: case 1. (*b*) Variation of the estimated diffusion coefficient with energy level: case 1. (*c*) Variation of the estimated drift coefficient with energy level: case 1. (*d*) The histogram for the energy: case 1.

which wave conditions remain approximately stationary (*ca.* 3 h). The input spectrum parameters were chosen to be $\beta = 1$ and $\omega_{\mathrm{p}} = 2$ and the excitation level was set so that a sample function of the indicated length could be generated such that large-amplitude rolling occurred, but without capsize. Figure 1 shows the correlation function of the excitation process, for the chosen parameters (noting that time here, and in the following, has been non-dimensionalized).

As pointed out earlier, in applying the proposed estimation technique it is first necessary to establish a suitable value for Δt. This can be achieved by establishing the range of Δt over which the estimates of drift and diffusion coefficients, calculated according to (5.2) and (5.3), are stable. Pilot studies revealed that, of the two coefficients, the diffusion coefficient was the most sensitive to the choice of Δt. According to (3.59), $D(E)$ becomes proportional to E, as E becomes small, since the restoring moment is nearly linear at low amplitudes. Thus $s_1 \to 1$ and $S_y[\omega(E)] \to S_y(1)$ as $E \to 0$. Thus it is convenient to examine the dependency of the slope $\hat{D}(E)/E$ at low E values, on the interval Δt. Figure 2*a* shows this variation and compares it with the theoretical value of $2\pi S_y(1)$. As expected from the theory, there is a range

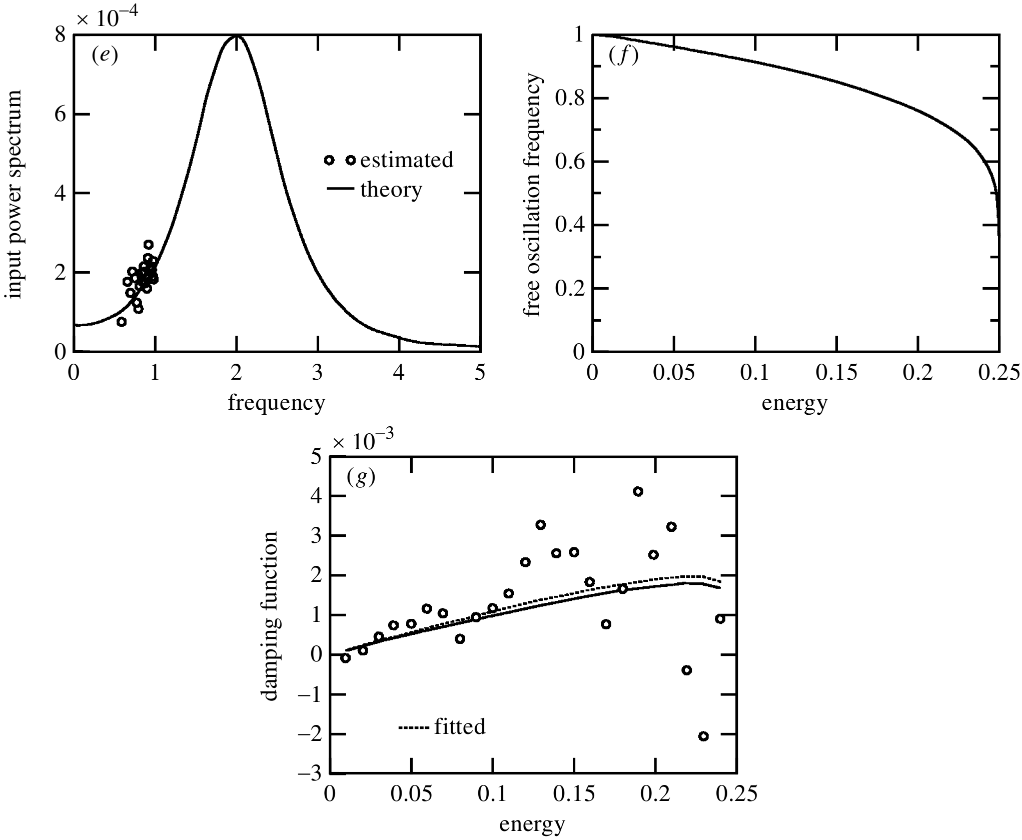

Figure 2. (*Cont.*) (*e*) Estimated input spectrum: case 1. (*f*) Variation of the undamped natural frequency with energy level. (*g*) Variation of the estimated damping function with energy level: case 1.

of Δt values ($1 < \Delta t < 10$) over which the slope is reasonably stable: for $\Delta t < 1$, the slope value drops rapidly, as E reduces, whereas for $\Delta t > 10$, it decays slowly. On the basis of this result, a value of Δt of 5 was chosen for calculating the drift and diffusion coefficient estimates.

Figure 2*b*, *c* shows, respectively, estimates of the diffusion and drift coefficients. The corresponding histogram of the energy values is shown in figure 2*d*. The value of E corresponding to capsize is 0.25. The histogram indicates that energy values close to this critical value occur in the data, but that $E < 0.15$ for most of the roll-response record. There is considerable scatter in the estimates of the two coefficients, but at low amplitudes, where there are most data, the agreement between the estimates and the theory is very good. A high level of scatter must be expected for this case since, when the damping is light, neighbouring roll-angle values in the time history are highly correlated and the effective number of independent sample values of the roll angle is much smaller than the actual number.

Figure 2*e* shows estimates of the input power spectrum, derived from the estimates of the diffusion coefficient, according to (5.4). This shows clearly that the theory correctly accounts for the shape of the input spectrum. As can be seen from a

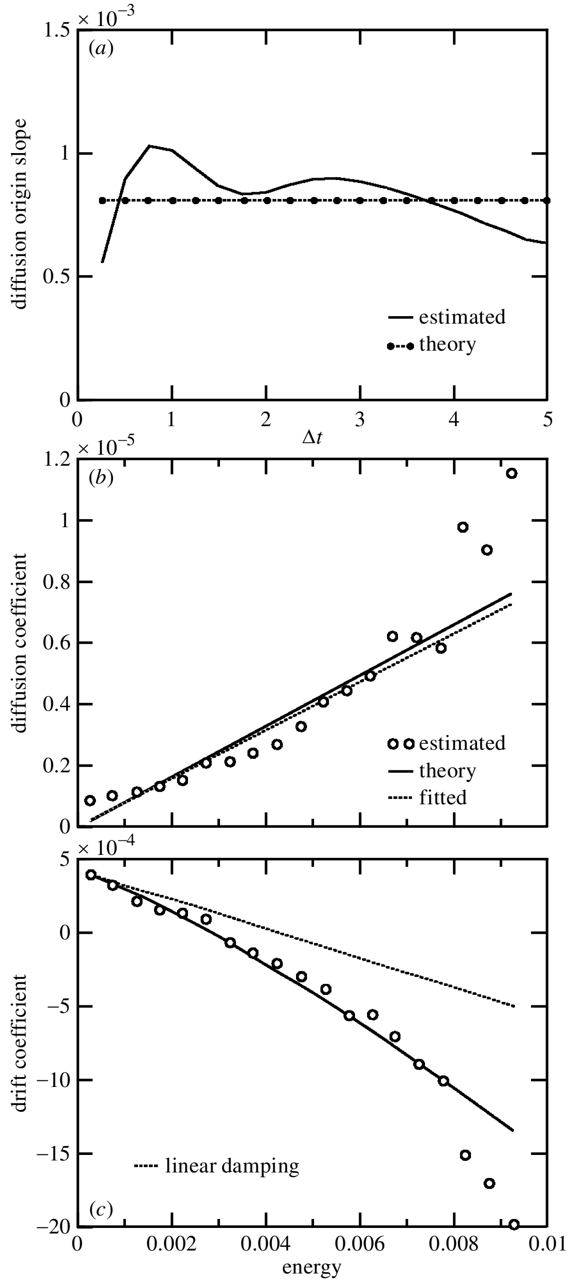

Figure 3. (a) Variation of the slope of the estimated diffusion coefficient with the interval Δt: case 2. (b) Variation of the estimated diffusion coefficient with energy level: case 2. (c) Variation of the estimated drift coefficient with energy level: case 2.

comparison of the energy histogram with a plot of the variation of the free oscillation frequency $\omega(E)$, with E (see figure $2f$), reliable estimates of the spectrum can only be achieved over a fairly limited frequency range (say $0.8 < \omega < 1$).

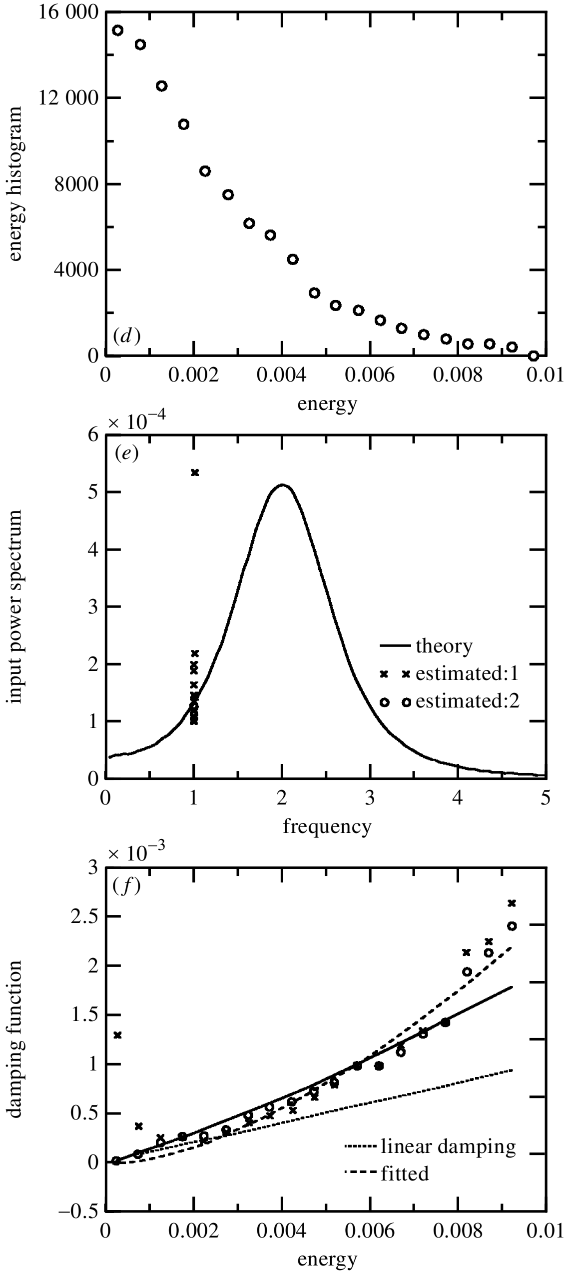

Figure 3. (*Cont.*) (*d*) The histogram for the energy: case 2. (*e*) Estimated input spectrum: case 2. (*f*) Variation of the estimated damping function with energy level: case 2.

Figure 2*g* shows the variation of the estimated damping function $d(E)$ with energy level. For comparison purposes, the theoretical variation is also shown, using the true linear damping value. This is given by, from (3.18),

$$d(E) = as_1^2 E. \tag{5.15}$$

Also shown is the result of fitting (5.15) to the damping function estimates. This was obtained by finding the value of a which minimizes the mean square difference between the estimates and the theoretical parametric form. There is a reasonably good agreement between the fitted curve, obtained in this way, and the true damping function.

The method was also tested on data relating to a higher level of damping ($a = 0.1$, $b = 0.8$), more typical of that encountered in practice (see, for example, Roberts 1985). For these data, referred to here as case 2, the other parameters were identical to those in case 1. The excitation level was chosen to give a moderate degree of response (maximum E about 0.01, corresponding to a maximum roll angle of *ca.* 15°: the restoring force was almost linear in this range). Figure 3*a* shows that, for this case, the slope of the diffusion coefficient is stable over a smaller range of Δt values than that for case 1. As expected from the theory, the stability range will reduce as the damping level increases. However, a reasonably stable range is obtained here ($1 < \Delta t < 4$) and a value of $\Delta t = 1.25$ was chosen to estimate the drift and diffusion coefficients.

Figure 3*b*, *c* shows, respectively, estimates of the diffusion and drift coefficients for this second case. The scatter is now much lower, reflecting the fact that, with higher damping levels, the effective number of independent samples increases, for a fixed length of data. There is a reasonably good agreement with the theoretical variations of these coefficients with energy, and figure 3*c* shows that the effect of the nonlinear damping term on the magnitude of the drift coefficient is significant. An inspection of the histogram of the energy values (figure 3*d*) shows that nearly all the data lie in the range $0 < E < 0.01$. Also shown in figure 3*b* is the least-squares fit to the data of the linear variation $D(E) = \alpha E$, where α is a constant. The close agreement with the theoretical variation confirms that, in this energy range, the restoring moment is effectively linear and that the approximation $D(E) = 2\pi S_y(1)E$ is valid. Thus the spectrum at $\omega = 1$ can be estimated from

$$\hat{S}_y(1) = \frac{\alpha}{2\pi}. \tag{5.16}$$

Figure 3*e* shows estimates of the input power spectrum, derived from (5.4) (labelled 'estimated: 1') and (5.16) (labelled 'estimated: 2'). The first set of estimates is very scattered, but the single estimate obtained from (5.16) is very close to the true value of $S_y(1)$.

Finally, figure 3*f* shows the variation of the estimates of the damping function $d(E)$ with energy level. These are compared with the theoretical variation, which is given by (from (3.16), truncating the Fourier expansion at the first term)

$$d(E) = as_1^2 E + 1.20bs_1^2 E^{3/2}. \tag{5.17}$$

Once again, the significant contribution of the nonlinear component of the damping is evident. The estimates of the damping function were obtained in two ways, both using (5.6). In the first way, $\hat{S}_y(\omega_j)$ was obtained from (5.4) (referred to as 'estimated: 1'), whereas in the second way the spectral estimate given by (5.16) was used (referred to as 'estimated: 2'). It can be seen that the first set of estimates has a significant error at low values of E, originating from the high relative error of the raw estimates of $D(E)$ in this range (see figure 3*b*). In contrast, the second set of estimates are in good agreement with the theoretical curve, except at the highest energy levels where the amount of data is relatively small. Also shown in figure 3f is the result of fitting the parametric form given by (5.17) to the second set of estimates: here the difference was minimized, in a least-squares sense, with respect to a and b. Although the damping parameters obtained in this way are not accurate, the damping function so obtained is in reasonable agreement with the true function.

6. Conclusions

The principal conclusions are summarized as follows.

(1) The roll motion of a ship has been treated as uncoupled from other motions, such that a single-degree-of-freedom nonlinear equation of motion, relating the roll angle to the roll moment, is appropriate. On this basis, it has been shown that, for light damping and non-white stochastic excitation, it is possible to model the energy envelope of the roll response as a one-dimensional continuous Markov process. Explicit expressions have been obtained for the drift and diffusion coefficients defining this process.

(2) If the mean time to capsize is very long, such that the response reaches stationarity in an approximate sense, then simple expressions for the probability distribution of the roll response have been obtained. Moreover, it has been shown that estimates of the mean time for the energy envelope to reach a specified critical level can be generated.

(3) A new method of estimating the damping and excitation moments has been proposed, based on estimating the drift and diffusion coefficients from roll-response data and using these estimates in conjunction with the theoretical expressions for these quantities. The method has been validated through an analysis of some digitally simulated data.

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